DAY THIRTY THREE

Unit Test 5

(Vectors and 3D Geometry)

1 The equation of plane perpendicular to $2x + 6y + 6z = 1$ and passing through the points $(2, 2, 1)$ and $(9, 3, 6)$, is

(a) $3x + 4y + 5z - 9 = 0$
(b) $3x + 4y - 5z + 9 = 0$
(c) $3x + 4y - 5z - 9 = 0$
(d) $3x + 4y + 5z + 9 = 0$

2 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be the unit vectors such that $\mathbf{a}$ and $\mathbf{b}$ are mutually perpendicular and $\mathbf{c}$ is equally inclined to $\mathbf{a}$ and $\mathbf{b}$ at an angle θ. If $\mathbf{c} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b})$, then

(a) $z^2 = 1 - 2x^2$
(b) $z^2 = 1 - x^2 + y^2$
(c) $z^2 = 1 + 2y^2$
(d) None of the above

3 If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{d}$ and $\mathbf{b} + \mathbf{c} + \mathbf{d} = \beta\mathbf{a}$, then $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$ is equal to

(a) $\mathbf{0}$
(b) $\alpha\mathbf{a}$
(c) β
(d) $(\alpha + \beta)\mathbf{c}$

4 The plane $ax + by = 0$ is rotated through an angle α about its line of intersection with the plane $z = 0$. Then, the equation to the plane in new position is

(a) $ax - by \pm z\sqrt{a^2 + b^2}\cot\alpha = 0$
(b) $ax + by \pm z\sqrt{a^2 + b^2}\cot\alpha = 0$
(c) $ax - by \pm z\sqrt{a^2 + b^2}\tan\alpha = 0$
(d) $ax + by \pm z\sqrt{a^2 + b^2}\tan\alpha = 0$

5 If the axes are rectangular, the distance from the point $(3, 4, 5)$ to the point, where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane $x + y + z = 17$ is

(a) 1
(b) 2
(c) 3
(d) None of these

6 Let $A(3, 2, 0)$, $B(5, 3, 2)$, $C(-9, 6, -3)$ are three points forming a triangle. If AD, the bisector of $\angle BAC$ meets BC in D, then coordinates of D are

(a) $\left(-\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$
(b) $\left(\frac{19}{8}, -\frac{57}{16}, \frac{17}{16}\right)$
(c) $\left(\frac{19}{8}, \frac{57}{16}, -\frac{17}{16}\right)$
(d) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

7 Vectors $\mathbf{a}$ and $\mathbf{b}$ are inclined at an angle $\theta = 120°$. If $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$, then $[(\mathbf{a} + 3\mathbf{b}) \times (3\mathbf{a} - \mathbf{b})]^2$ is equal to

(a) 300 (b) 325 (c) 275 (d) 225

8 A point moves so that the sum of the squares of its distances from the six faces of a cube given by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$ is 10 units. The locus of the point is

(a) $x^2 + y^2 + z^2 = 1$
(b) $x^2 + y^2 + z^2 = 2$
(c) $x + y + z = 1$
(d) $x + y + z = 2$

9 If $(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = 144$ and $|\mathbf{a}| = 4$, then $|\mathbf{b}|$ is equal to

(a) 3 (b) 8 (c) 12 (d) 16

10 The values of x for which the angle between $\mathbf{a} = 2x^2\mathbf{i} + 4x\mathbf{j} + \mathbf{k}$, $\mathbf{b} = 7\mathbf{i} - 2\mathbf{j} + x\mathbf{k}$ is obtuse, is

(a) $x > 1/2$ or $x < 0$
(b) $0 < x < 1/2$
(c) $1/2 < x < 15$
(d) None of these

11 Let $\mathbf{a} = 3\mathbf{i} + 2\mathbf{k}$ and $\mathbf{b} = 2\mathbf{j} + \mathbf{k}$. If $\mathbf{c}$ is a unit vector, then the maximum value of the vector triple product $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$, is

(a) $\sqrt{61}$
(b) $\sqrt{59}$
(c) $\sqrt{3} \cdot \sqrt{36}$
(d) None of these

12 The ratio of lengths of diagonals of the parallelogram constructed on the vectors $\mathbf{a} = 3\mathbf{p} - \mathbf{q}$, $\mathbf{b} = \mathbf{p} + 3\mathbf{q}$ is (given that $|\mathbf{p}| = |\mathbf{q}| = 2$ and the angle between $\mathbf{p}$ and $\mathbf{q}$ is $\frac{\pi}{3}$)

(a) $\sqrt{7} : \sqrt{3}$ (b) $\sqrt{3} : \sqrt{2}$ (c) $\sqrt{5} : \sqrt{7}$ (d) $\sqrt{5} : \sqrt{3}$

13 Let **p**, **q** and **r** be three mutually perpendicular vectors of the same magnitude. If a vector x satisfies equation
$\mathbf{p}\times\{(\mathbf{x}-\mathbf{q})\times\mathbf{p}\}+\mathbf{q}\times\{(\mathbf{x}-\mathbf{r})\times\mathbf{q}\}+\mathbf{r}\times\{(\mathbf{x}-\mathbf{p})\times\mathbf{r}\}=\mathbf{0}$
Then, **x** is given by
(a) $\frac{1}{2}(\mathbf{p}+\mathbf{q}-2\mathbf{r})$ (b) $\frac{1}{2}(\mathbf{p}+\mathbf{q}+\mathbf{r})$
(c) $\frac{1}{3}(2\mathbf{p}+\mathbf{q}-\mathbf{r})$ (d) None of these

14 A line with direction cosines proportional to 2, 1, 2 meets each of the lines given by the equation $x=y+2=z$; $x+2=2y=2z$.
The coordinates of the point of intersection are
(a) (6, 4, 6), (2, 4, 2) (b) (6, 6, 6), (2, 6, 2)
(c) (6, 4, 6), (2, 2, 0) (d) None of these

15 The vector **B** satisfying the vector equation $\mathbf{A}+\mathbf{B}=\mathbf{a}$, $\mathbf{A}\times\mathbf{B}=\mathbf{b}$ and $\mathbf{A}\cdot\mathbf{a}=1$, where **a** and **b** are given vectors is
(a) $\frac{(\mathbf{b}\times\mathbf{a})+\mathbf{a}\,(a^2-1)}{a^2}$ (b) $\frac{(\mathbf{a}\times\mathbf{b})+\mathbf{a}}{\mathbf{a}}$
(c) $\frac{\mathbf{a}\,(a^2-1)+\mathbf{b}(b^2-1)}{a^2}$ (d) None of these

16 The vector **c**, directed along the bisectors of the angle between the vectors $\mathbf{a}=7\mathbf{i}-4\mathbf{j}-4\mathbf{k}$, $\mathbf{b}=-2\mathbf{i}-\mathbf{j}+2\mathbf{k}$ and $|\mathbf{c}|=5\sqrt{6}$ is
(a) $\pm\frac{5}{3}(\mathbf{i}-7\mathbf{j}+2\mathbf{k})$ (b) $\pm\frac{5}{3}(5\mathbf{i}+5\mathbf{j}+2\mathbf{k})$
(c) $\pm\frac{5}{3}(\mathbf{i}+7\mathbf{j}+2\mathbf{k})$ (d) $\pm\frac{5}{3}(-5\mathbf{i}+5\mathbf{j}+2\mathbf{k})$

17 Let **a**, **b** and **c** be three non-zero and non-coplanar vectors and **p**, **q** and **r** be three vectors given by $\mathbf{p}=\mathbf{a}+\mathbf{b}-2\mathbf{c}$, $\mathbf{q}=3\mathbf{a}-2\mathbf{b}+\mathbf{c}$ and $\mathbf{r}=\mathbf{a}-4\mathbf{b}+2\mathbf{c}$. If the volume of the parallelopiped determined by **a**, **b** and **c** is V_1 and the volume of the parallelopiped determined by **p**, **q** and **r** is V_2, then $V_2:V_1$ is equal to
(a) 7 : 1 (b) 3 : 1
(c) 11 : 1 (d) None of these

18 A non-zero vector **a** is parallel to the line of intersection of the plane determined by the vectors **i**, **i** + **j** and the plane determined by the vectors **i** – **j**, **i** + **k**. Then, the angle between **a** and the vector **i** – 2**j** + 2**k** is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

19 Let **a**, **b** and **c** be three vectors having magnitudes 1, 1 and 2, respectively. If $\mathbf{a}\times(\mathbf{a}\times\mathbf{c})+\mathbf{b}=\mathbf{0}$, then the acute angle between **a** and **c** is
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) None of these

20 If **a**, **b** and **c** are unit coplanar vectors, then the scalar triple product $[2\mathbf{a}-\mathbf{b}\;\;2\mathbf{b}-\mathbf{c}\;\;2\mathbf{c}-\mathbf{a}]$ is equal to
(a) 0 (b) 1 (c) $-\sqrt{3}$ (d) $\sqrt{3}$

21 Point (α,β,γ) lies on the plane $x+y+z=2$. Let $\mathbf{a}=\alpha\,\mathbf{i}+\beta\,\mathbf{j}+\gamma\,\mathbf{k}$, $\mathbf{k}\times(\mathbf{k}\times\mathbf{a})=\mathbf{0}$. Then, γ is equal to
(a) 0 (b) 1
(c) 2 (d) $\frac{1}{2}$

22 If the four plane faces of a tetrahedron are represented by the equation $\mathbf{r}\cdot(l\,\mathbf{i}+m\,\mathbf{j})=0$, $\mathbf{r}\cdot(m\,\mathbf{j}+n\,\mathbf{k})=0$, $\mathbf{r}\cdot(n\mathbf{k}+p\mathbf{i})=0$ and $\mathbf{r}\cdot(l\,\mathbf{i}+m\,\mathbf{j}+n\mathbf{k})=p$, then the volume of the tetrahedron is
(a) $\frac{p^3}{6\,lmn}$ (b) $\frac{2p^3}{3\,lmn}$
(c) $\frac{3\,p^3}{lmn}$ (d) $\frac{6p^3}{lmn}$

23 If a variable plane forms a tetrahedron of constant volume $64\,k^3$ with the coordinate planes, then the locus of the centroid of the tetrahedron is
(a) $xyz=k^3$ (b) $xyz=2k^3$
(c) $xyz=12k^3$ (d) $xyz=6k^3$

24 The line through $\mathbf{i}+3\mathbf{j}+2\mathbf{k}$ and perpendicular to the line $\mathbf{r}=(\mathbf{i}+2\mathbf{j}-\mathbf{k})+\lambda\,(2\mathbf{i}+\mathbf{j}+\mathbf{k})$ and $\mathbf{r}=(2\mathbf{i}+6\mathbf{j}+\mathbf{k})+\mu\,(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$ is
(a) $\mathbf{r}=(\mathbf{i}+2\mathbf{j}-\mathbf{k})+\lambda\,(-\mathbf{i}+5\mathbf{j}-3\mathbf{k})$
(b) $\mathbf{r}=(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+\lambda\,(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$
(c) $\mathbf{r}=(\mathbf{i}+3\mathbf{j}+2\mathbf{k})+\lambda\,(\mathbf{i}+5\mathbf{j}+3\mathbf{k})$
(d) None of the above

25 The orthogonal projection A' of the point A with position vector (1, 2, 3) on the plane $3x-y+4z=0$, is
(a) (–1, 3, –1) (b) $\left(-\frac{1}{2},\frac{5}{2},1\right)$
(c) $\left(\frac{1}{2},-\frac{5}{2},-1\right)$ (d) (6, –7, –5)

26 The equation of the plane containing the points $A(1,0,1)$ and $B(3,1,2)$ and parallel to the line joining the origin to the point $C(1,-1,2)$ is
(a) $x+y-z=0$ (b) $x+y+z=0$
(c) $x-y+z=0$ (d) $x-y-z=0$

27 The planes $3x-y+z+1=0$ and $5x+y+3z=0$ intersect in the line PQ. The equation of the plane through the point (2, 1, 4) and perpendicular to PQ is
(a) $x+y-2z=5$ (b) $x+y-2z=-5$
(c) $x+y+2z=5$ (d) $x+y+2z=-5$

28 The line of intersection of the planes $\mathbf{r}\cdot(3\mathbf{i}-\mathbf{j}+\mathbf{k})=1$ and $\mathbf{r}\cdot(\mathbf{i}+4\mathbf{j}-2\mathbf{k})=2$ is parallel to the vector
(a) $-2\mathbf{i}+7\mathbf{j}+13\mathbf{k}$ (b) $-2\mathbf{i}-7\mathbf{j}+13\mathbf{k}$
(c) $2\mathbf{i}+7\mathbf{j}-13\mathbf{k}$ (d) None of these

29 A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x=y+a=z$ and $x+a=2y=2z$. The coordinates of each of the points of intersection are given by
(a) $(3a, 2a, 3a), (a, a, 2a)$ (b) $(3a, 3a, 3a), (a, a, a)$
(c) $(3a, 2a, 3a), (a, a, a)$ (d) None of these

30 The sides of a parallelogram are $3\mathbf{i} + 4\mathbf{j} - 6\mathbf{k}$ and $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$. The unit vector parallel to one of the diagonals is

(a) $\frac{5\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{27}}$ (b) $\frac{4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}}{\sqrt{29}}$

(c) $\frac{\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}}{\sqrt{21}}$ (d) None of these

31 Let $\mathbf{p} = 8\mathbf{i} + 6\mathbf{j}$ and $\mathbf{q}$ be two vectors perpendicular to each other in the xy-plane. Then, the vector in the same plane having projections 2 and 4 along $\mathbf{p}$ and $\mathbf{q}$ respectively is

(a) $\pm 3(\mathbf{i} - 2\mathbf{j})$ (b) $\pm(\mathbf{i} + 2\mathbf{j})$

(c) $\pm 2(2\mathbf{i} - \mathbf{j})$ (d) None of these

32 The equation of the plane containing the line $2x - y + z - 3 = 0$ and $3x + y + z = 5$ at a distance of $\frac{1}{\sqrt{6}}$ from the point $(2, 1, -1)$ is

(a) $x + y + z - 3 = 0$

(b) $2x - y - z - 3 = 0$

(c) $2x - y + z + 3 = 0$

(d) $62x + 29y + 19z - 105 = 0$

33 The equation of the plane through the point $(2, -1, -3)$ and parallel to the lines $\frac{x-1}{3} = \frac{y+2}{2} = \frac{z}{-4}$ and $\frac{x}{2} = \frac{y-1}{-3} = \frac{z-2}{2}$ is

(a) $8x + 14y + 13z + 37 = 0$ (b) $8x - 14y + 13y + 37 = 0$

(c) $8x + 14y - 13z + 37 = 0$ (d) None of these

34 The shortest distance between the lines

$\mathbf{r} = -(\mathbf{i} + \mathbf{j} + \mathbf{k}) + \lambda(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$

and $\mathbf{r} = -\mathbf{i} + \mu(3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k})$ is

(a) 1 (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

35 A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. The angle between the faces OAB and ABC is

(a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{17}{31}\right)$

(c) 30° (d) 90°

36 If $\mathbf{p} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$ and $\mathbf{q} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ be two vectors and $\mathbf{r}$ is a vector perpendicular to $\mathbf{p}$ and $\mathbf{q}$ and satisfying the condition. $\mathbf{r}(2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = -12$, then $\mathbf{r}$ is equal to

(a) $2\mathbf{i} - \frac{20}{3}\mathbf{j} + 16\mathbf{k}$ (b) $\frac{2}{3}(3\mathbf{i} + 10\mathbf{j} + 8\mathbf{k})$

(c) $\frac{1}{3}(3\mathbf{i} - 10\mathbf{j} + 8\mathbf{k})$ (d) None of these

37 The direction ratios of a normal to the plane through (2, 0, 0) and (0, 2, 0) that makes an angle $\frac{\pi}{3}$ with the plane $2x + 3y = 5$ is

(a) 1 : 1 : 2 (b) $1 : 1 : \sqrt{3}$

(c) $\sqrt{2} : 1 : 3$ (d) $1 : 1 : \sqrt{5.7}$

38 The shortest distance between the lines $\frac{x-2}{3} = \frac{y-1}{2} = \frac{z-3}{2}$ and $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-4}{4}$ is equal to

(a) 1.14 units (b) 2.01 units

(c) 3.16 units (d) None of these

39 The intersecting point of lines, $L_1 = \frac{x+1}{-3} = \frac{y-3}{2} = \frac{y+2}{1}$ and $L_2 = \frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ is

(a) $(-3, 2, 1)$ (b) $(2, 1, -3)$

(c) $(1, -3, 2)$ (d) None of these

40 If $\mathbf{a}$ and $\mathbf{b}$ are unit vectos, then the greatest value of $|\mathbf{a} + \mathbf{b}| + |\mathbf{a} - \mathbf{b}|$ is

(a) 2 (b) 4 (c) $2\sqrt{2}$ (d) $\sqrt{2}$

41 If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and r is a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are non-coplanar for

(a) no value of λ (b) all except one value of λ

(c) all except two values of λ (d) all values of λ

42 Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that $\mathbf{a}$ is perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$. If the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\frac{\pi}{3}$, then $|\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}|$ is equal to

(a) 1/3 (b) 1/2 (c) 1 (d) 2

43 The distance of the point $A(-2, 3, 1)$ from the line PQ through $P(-3, 5, 2)$ which make equal angles with the axes is

(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{\frac{14}{3}}$ (c) $\frac{16}{\sqrt{3}}$ (d) $\frac{5}{\sqrt{3}}$

44 The plane passing through the point (5, 1, 2) perpendicular to the line $2(x-2) = y - 4 = z - 5$ will meet the line in the point

(a) (1, 2, 3) (b) (2, 3, 1)

(c) (1, 3, 2) (d) (3, 2, 1)

Direction (Q. Nos. 45-48) *Each of these questions contains two statements : Statement I (Assertion) and Statement II (Reason). Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.*

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true; Statement II is false

(d) Statement I is false; Statement II is true

45 Consider vectors $\mathbf{a}$ and $\mathbf{c}$ are non-collinear, then

Statement I The lines $\mathbf{r} = 6\mathbf{a} - \mathbf{c} + \lambda(2\mathbf{c} - \mathbf{a})$ and $\mathbf{r} = \mathbf{a} - \mathbf{c} + \mu(\mathbf{a} + 3\mathbf{c})$ are coplanar.

Statement II There exist λ and μ such that the two values of r become same.

46 Consider **u** and **v** are unit vectors inclined at an angle α and **a** is a unit vector bisecting the angle between them,

Statement I Then, $\mathbf{a} = \dfrac{\mathbf{u} + \mathbf{v}}{2\cos(\alpha/2)}$.

Statement II If ABC is an isosceles triangle with $AB = AC = 1$, then vector representing bisector of $\angle A$ is $\dfrac{\mathbf{AB} + \mathbf{AC}}{2}$.

47 Suppose $\pi : x + y - 2z = 3, P : (2, 1, 6), Q : (6, 5, -2)$

Statement I The line joining PQ is perpendicular to the normal to the plane π.

Statement II Q is the image of P in the plane π.

48 Consider the equation of planes $P_1 = x + y + z - 6 = 0$ and $P_2 = 2x + 3y + 4z + 5 = 0$.

Statement I The equation of the plane through the intersection of the planes P_1 and P_2 and the point $(4, 4, 4)$ is $29x + 23y + 17y = 276$.

Statement II Equation of the plane through the line of intersection of the planes $P_1 = 0$ and $P_2 = 0$ is $P_1 + \lambda P_2 = 0$, $\lambda \neq 0$.

49 The two adjacent sides of a parallelogram are $2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$.

Statement I The unit vector parallel to its diagonal is $\frac{3}{5}\mathbf{i} - \frac{6}{5}\mathbf{j} + \frac{2}{5}\mathbf{k}$.

Statement II Area of parallelogram is $11\sqrt{5}$ sq units.

(a) Only Statement I is true (b) Only Statement II is true
(c) Both statements are ture (d) Both statements are false

ANSWERS

1. (c)	**2.** (a)	**3.** (a)	**4.** (d)	**5.** (c)	**6.** (d)	**7.** (a)	**8.** (b)	**9.** (a)	**10.** (b)
11. (a)	**12.** (a)	**13.** (b)	**14.** (d)	**15.** (a)	**16.** (a)	**17.** (d)	**18.** (b)	**19.** (a)	**20.** (a)
21. (c)	**22.** (b)	**23.** (d)	**24.** (b)	**25.** (b)	**26.** (d)	**27.** (b)	**28.** (a)	**29.** (c)	**30.** (a)
31. (c)	**32.** (d)	**33.** (a)	**34.** (d)	**35.** (a)	**36.** (d)	**37.** (d)	**38.** (d)	**39.** (b)	**40.** (c)
41. (c)	**42.** (c)	**43.** (b)	**44.** (a)	**45.** (a)	**46.** (a)	**47.** (d)	**48.** (a)	**49.** (b)	

Hints and Explanations

1 The plane passing through $(2, 2, 1)$, is

$$a(x - 2) + b(y - 2) + c(z - 1) = 0$$

Since, it passes through $(9, 3, 6)$.

$\therefore \quad 7a + b + 5c = 0 \quad \ldots(i)$

Since, it is perpendicular to $2x + 6y + 6z - 1 = 0$.

$\therefore \quad 2a + 6b + 6c = 0 \quad \ldots(ii)$

$\Rightarrow \quad \dfrac{a}{-24} = \dfrac{b}{-32} = \dfrac{c}{40}$ [from Eqs. (i) and (ii)]

$\Rightarrow \quad \dfrac{a}{3} = \dfrac{b}{4} = \dfrac{c}{-5}$

The required plane is

$$3(x - 2) + 4(y - 2) - 5(z - 1) = 0$$

$\Rightarrow \quad 3x + 4y - 5z - 9 = 0$

2 Now, $\mathbf{a} \cdot \mathbf{c} = x(\mathbf{a} \cdot \mathbf{a}) + y(\mathbf{a} \cdot \mathbf{b}) + z\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$

$\Rightarrow \quad x = \cos\theta$

Similarly, $y = \cos\theta$

Now, $|\mathbf{c}|^2 = x^2|\mathbf{a}|^2 + y^2|\mathbf{b}|^2 + z^2|\mathbf{a} \times \mathbf{b}|^2$

$\Rightarrow \quad 1 - 2\cos^2\theta = z^2 \Rightarrow 1 - 2x^2 = z^2$

where, $z = |\mathbf{a}\ \mathbf{b}\ \mathbf{c}|$

3 Given, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{d}$ and $\mathbf{b} + \mathbf{c} + \mathbf{d} = \beta\mathbf{a}$

$\therefore \quad \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = (\alpha + 1)\mathbf{d}$

and $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = (\beta + 1)\mathbf{a}$

$\Rightarrow \quad (\alpha + 1)\mathbf{d} = (\beta + 1)\mathbf{a}$

If $\alpha \neq -1$, then $(\alpha + 1)\mathbf{d} = (\beta + 1)\mathbf{a}$

$\Rightarrow \quad \mathbf{d} = \dfrac{\beta + 1}{\alpha + 1}\mathbf{a}$

$\therefore \quad \mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\left(\dfrac{\beta + 1}{\alpha + 1}\right)\mathbf{a}$

$\Rightarrow \quad \left(1 - \dfrac{\alpha(\beta + 1)}{\alpha + 1}\right)\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$

Hence, **a**, **b** and **c** are coplanar, which is a contradiction to the given condition.

$\therefore \quad \alpha = -1$

$\Rightarrow \quad \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$

4 Equation of any plane passing through the line of intersection of given plane, is

$$ax + by + kz = 0 \quad \ldots(i)$$

$\therefore$ DC's of Eq. (i) are

$$\frac{a}{\sqrt{a^2 + b^2 + k^2}}, \frac{b}{\sqrt{a^2 + b^2 + k^2}}, \frac{k}{\sqrt{a^2 + b^2 + k^2}}$$

The DC's of a normal to the given plane is

$$\frac{a}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}}, 0.$$

$\therefore \quad \cos\alpha = \dfrac{a \cdot a + b \cdot b + k \cdot 0}{\sqrt{a^2 + b^2 + k^2}\sqrt{a^2 + b^2}}$

$= \sqrt{\dfrac{a^2 + b^2}{a^2 + b^2 + k^2}}$

$\Rightarrow \quad k^2\cos^2\alpha = a^2(1 - \cos^2\alpha) + b^2(1 - \cos^2\alpha)$

$\Rightarrow \quad k^2 = \dfrac{(a^2 + b^2)\sin^2\alpha}{\cos^2\alpha}$

$\therefore \quad k = \pm\sqrt{a^2 + b^2}\tan\alpha$

From Eq. (i),

$$ax + by \pm z\sqrt{a^2 + b^2}\tan\alpha = 0$$

5 Any point on the line is $(r + 3, 2r + 4, 2r + 5)$ which lies on the plane $x + y + z = 17$.

$\because \quad (r + 3) + (2r + 4) + (2r + 5) = 17$

$\therefore \quad r = 1$

Thus, the point of intersection is $(4, 6, 7)$. So, the required distance

$= \sqrt{(4-3)^2 + (6-4)^2 + (7-5)^2}$

$= \sqrt{1+4+4} = 3$

6 Here, $AB = \sqrt{(5-3)^2 + (3-2)^2 + (2-0)^2} = 3$

and $AC = \sqrt{(-9-3)^2 + (6-2)^2 + (-3-0)^2} = 13$

Since, AD is the bisector of $\angle BAC$.

$\therefore \quad \frac{BD}{DC} = \frac{AB}{AC} = \frac{3}{13}$

Since, D divides BC in the ratio $3:13$.

$\therefore$ The coordinates of D are

$$\left[\frac{3(-9)+13(5)}{3+13}, \frac{3(6)+13(3)}{3+13}, \frac{3(-3)+13(2)}{3+13}\right]$$

$$= \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$$

7 $[(\mathbf{a}+3\mathbf{b})\times(3\mathbf{a}-\mathbf{b})]^2 = [10(\mathbf{b}\times\mathbf{a})]^2$

$= 100[|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2]$

$= 100[1\times 4 - (1\times 2\times \cos 120°)^2]$

$= 100(4-1) = 300$

8 Let $P(x, y, z)$ be any point on the locus, then the distances from the six faces are $|x+1|$, $|x-1|$, $|y+1|$, $|y-1|$, $|z+1|$, $|z-1|$

According to the given condition,

$$|x+1|^2 + |x-1|^2 + |y+1|^2 + |y-1|^2 + |z+1|^2 + |z-1|^2 = 10$$

$\Rightarrow \quad 2(x^2+y^2+z^2) = 10 - 6 = 4$

$\therefore \quad x^2+y^2+z^2 = 2$

9 $(\mathbf{a}\times\mathbf{b})^2 + (\mathbf{a}\cdot\mathbf{b})^2 = |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta + |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta$

$= |\mathbf{a}|^2|\mathbf{b}|^2$

$\Rightarrow \quad 144 = (4)^2|\mathbf{b}|^2 \Rightarrow |\mathbf{b}| = 3$

10 Since, $\quad \mathbf{a}\cdot\mathbf{b} < 0$

$\Rightarrow \quad 14x^2 - 8x + x < 0$

$\Rightarrow \quad 14x^2 - 7x < 0$

$\Rightarrow \quad 7x(2x-1) < 0$

$\therefore \quad 0 < x < \frac{1}{2}$

11 Here, $\mathbf{a}\times\mathbf{b} = -4\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$

Now, $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

$= (-4\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})\cdot\mathbf{c}$

$\therefore$ The maximum value of $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= |-4\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}||\mathbf{c}| = \sqrt{61}$

12 Now, $\mathbf{d}_1 = \mathbf{a} + \mathbf{b} = 4\mathbf{p} + 2\mathbf{q}$

and $\mathbf{d}_2 = \mathbf{a} - \mathbf{b} = 2\mathbf{p} - 4\mathbf{q}$

$\Rightarrow \mathbf{d}_1^2 = 16\mathbf{p}^2 + 4\mathbf{q}^2 + 16\mathbf{p}\cdot\mathbf{q}$

$= 16(4) + 4(4) + 16\left(2\times 2\times\cos\frac{\pi}{3}\right)$

$= 112$

$\Rightarrow \quad |\mathbf{d}_1| = 4\sqrt{7}$

Similarly, $|\mathbf{d}_2| = 4\sqrt{3}$

$\therefore \quad \mathbf{d}_1 : \mathbf{d}_2 = \sqrt{7} : \sqrt{3}$

13 $|\mathbf{p}| = |\mathbf{q}| = |\mathbf{r}| = c$ [say]

and $\mathbf{p}\cdot\mathbf{q} = 0 = \mathbf{p}\cdot\mathbf{r} = \mathbf{q}\cdot\mathbf{r}$

Given that,

$$\mathbf{p}\times\{(\mathbf{x}-\mathbf{q})\times\mathbf{p}\} + \mathbf{q}\times\{(\mathbf{x}-\mathbf{r})\times\mathbf{q}\} + \mathbf{r}\times\{(\mathbf{x}-\mathbf{p})\times\mathbf{r}\} = \mathbf{0}$$

$\Rightarrow (\mathbf{p}\cdot\mathbf{p})(\mathbf{x}-\mathbf{q}) - \{\mathbf{p}\cdot(\mathbf{x}-\mathbf{q})\}\mathbf{p} + \ldots = 0$

$\Rightarrow c^2(\mathbf{x}-\mathbf{q}+\mathbf{x}-\mathbf{r}+\mathbf{x}-\mathbf{p}) - (\mathbf{p}\cdot\mathbf{x})\mathbf{p} - (\mathbf{q}\cdot\mathbf{x})\mathbf{q} - (\mathbf{r}\cdot\mathbf{x})\mathbf{r} = 0$

$\Rightarrow c^2\{3\mathbf{x} - (\mathbf{p}+\mathbf{q}+\mathbf{r})\} - \{(\mathbf{p}\cdot\mathbf{x})\mathbf{p} + (\mathbf{q}\cdot\mathbf{x})\mathbf{q} + (\mathbf{r}\cdot\mathbf{x})\mathbf{r}\} = 0$

which is satisfied by $\mathbf{x} = \frac{1}{2}(\mathbf{p}+\mathbf{q}+\mathbf{r})$

14 Let $P(r, r-2, r)$ and $Q(2k-2, k, k)$ are the general coordinates of points on the two given lines.

$\therefore$ DR's of PQ are

$(r-2k+2, r-k-2, r-k)$

$\therefore \quad \frac{r-2k+2}{2} = \frac{r-k-2}{1} = \frac{r-k}{2}$

$\Rightarrow \quad r = 6, k = 2$

So, the points of intersection are $(6, 4, 6)$ and $(2, 2, 2)$.

15 Given, $\quad \mathbf{A} + \mathbf{B} = \mathbf{a}$ ...(i)

$\Rightarrow \quad \mathbf{A}\cdot\mathbf{a} + \mathbf{B}\cdot\mathbf{a} = \mathbf{a}\cdot\mathbf{a}$

$\Rightarrow \quad 1 + \mathbf{B}\cdot\mathbf{a} = a^2$

$\Rightarrow \quad \mathbf{B}\cdot\mathbf{a} = a^2 - 1$ ...(ii)

Also, $\quad \mathbf{A}\times\mathbf{B} = \mathbf{b}$

$\Rightarrow \quad \mathbf{a}\times(\mathbf{A}\times\mathbf{B}) = \mathbf{a}\times\mathbf{b}$

$\Rightarrow \quad (\mathbf{a}\cdot\mathbf{B})\mathbf{A} - (\mathbf{a}\cdot\mathbf{A})\mathbf{B} = \mathbf{a}\times\mathbf{b}$

$\Rightarrow \quad (a^2-1)\mathbf{A} - \mathbf{B} = \mathbf{a}\times\mathbf{b}$

[from Eq. (ii)] ...(iii)

From Eqs. (i) and (iii), we get

$$\mathbf{A} = \frac{(\mathbf{a}\times\mathbf{b}) + \mathbf{a}}{a^2}$$

and $\quad \mathbf{B} = \mathbf{a} - \left\{\frac{(\mathbf{a}\times\mathbf{b}) + \mathbf{a}}{a^2}\right\}$

$\therefore \quad \mathbf{B} = \frac{(\mathbf{b}\times\mathbf{a}) + \mathbf{a}(a^2-1)}{a^2}$

16 The required vector $\mathbf{c}$ is given by

$$\mathbf{c} = \pm\lambda\left(\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}\right)$$

$$= \pm\lambda\left\{\frac{1}{9}(7\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}) + \frac{1}{3}(-2\mathbf{i} - \mathbf{j} + 2\mathbf{k})\right\}$$

$\Rightarrow \quad \mathbf{c} = \pm\frac{\lambda}{9}(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

$\Rightarrow \quad 5\sqrt{6} = \frac{\lambda}{9}\sqrt{1+49+4}$

$= \frac{\lambda}{9}\sqrt{54} \quad [\because |\mathbf{c}| = 5\sqrt{6}]$

$\Rightarrow \quad \lambda = 15$

$\therefore \quad \mathbf{c} = \pm\frac{15}{9}(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

$= \pm\frac{5}{3}(\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

17 Given, $V_1 = [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$ and $V_2 = [\mathbf{p}\ \mathbf{q}\ \mathbf{r}]$

Then, $V_2 = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 1 & -4 & 2 \end{vmatrix}[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= 15[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$\therefore \quad V_2 : V_1 = 15:1$

18 The normal to the first plane is along $\mathbf{i}\times(\mathbf{i}+\mathbf{j}) = \mathbf{k}$ and the normal to the second plane is along

$$(\mathbf{i}-\mathbf{j})\times(\mathbf{i}+\mathbf{k}) = -\mathbf{i} - \mathbf{j} + \mathbf{k}$$

Since, $\mathbf{a}$ is perpendicular to the two normals.

So, $\mathbf{a}$ is along $\mathbf{k}\times(-\mathbf{i}-\mathbf{j}+\mathbf{k}) = \mathbf{i} - \mathbf{j}$

Hence, the angle between $\mathbf{a}$ and the vector $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is

$$\cos^{-1}\frac{\mathbf{a}\cdot(\mathbf{i}-2\mathbf{j}+2\mathbf{k})}{|\mathbf{a}||\mathbf{i}-2\mathbf{j}+2\mathbf{k}|}$$

$$= \cos^{-1}\frac{(\mathbf{i}-\mathbf{j})}{\sqrt{2}}\cdot\frac{(\mathbf{i}-2\mathbf{j}+2\mathbf{k})}{3}$$

$$= \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$$

19 Since, $\mathbf{b} = (\mathbf{a}\times\mathbf{c})\times\mathbf{a}$

$\Rightarrow \quad |\mathbf{b}| = |\mathbf{a}\times\mathbf{c}||\mathbf{a}| \Rightarrow 1 = 2\sin\theta$

$\therefore \quad \theta = \frac{\pi}{6}$

20 If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ lie in a plane, then $2\mathbf{a}-\mathbf{b}$, $2\mathbf{b}-\mathbf{c}$ and $2\mathbf{c}-\mathbf{a}$, also lie in the same plane. So, their scalar triple product is '0'.

21 Since, $\quad \mathbf{k}\times(\mathbf{k}\times\mathbf{a}) = 0$

$\Rightarrow \quad (\alpha\mathbf{i} + \beta\mathbf{j}) = 0$

$\Rightarrow \quad \alpha = \beta = 0$

$\because \quad \alpha + \beta + \gamma = 2$

$\therefore \quad \gamma = 2$

22 The first three planes meet at the point whose position vector is $(0, 0, 0)$. The first two and the fourth planes meet at the point whose position vector is $\left(\frac{p}{l}, \frac{-p}{m}, \frac{p}{n}\right)$. Similarly, the other two vertices of the tetrahedron have position vectors

$$\left(\frac{-p}{l}, \frac{p}{m}, \frac{p}{n}\right) \text{ and } \left(\frac{p}{l}, \frac{p}{m}, \frac{-p}{n}\right).$$

$\therefore$ Volume of the tetrahedron

$$= \frac{1}{6}\left\|\begin{matrix} p/l & -p/m & p/n \\ -p/l & p/m & p/n \\ p/l & p/m & -p/n \end{matrix}\right\|$$

$$= \frac{p^3}{6\,lmn}\left\|\begin{matrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{matrix}\right\|$$

$$= \frac{p^3}{6\,lmn}\left\|\begin{matrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & -1 \end{matrix}\right\| \quad \begin{bmatrix} R_1 \to R_1 + R_2, \\ R_2 \to R_2 + R_3 \end{bmatrix}$$

$$= \frac{|-4|p^3}{6\,lmn} = \frac{2p^3}{3\,lmn}$$

23 Let the variable plane intersects the coordinate axes at $A\ (a, 0, 0)$, $B\ (0, b, 0)$ and $C\ (0, 0, c)$.

Then, the equation of the plane will be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \ldots(i)$$

Let $P\ (\alpha, \beta, \gamma)$ be the centroid of tetrahedron $OABC$, then

$$\alpha = \frac{a}{4}, \beta = \frac{b}{4} \text{ and } \gamma = \frac{c}{4}$$

$\therefore \quad a = 4\alpha, b = 4\beta, c = 4\gamma$

Now, volume of tetrahedron

$= \text{(Area of } \Delta AOB) OC$

$\Rightarrow \quad 64\,k^3 = \frac{1}{3}\left(\frac{1}{2}\,ab\right)c = \frac{abc}{6}$

$\Rightarrow \quad 64\,k^3 = \frac{(4\alpha)(4\beta)\times(4\gamma)}{3\times 2}$

$\therefore \quad k^3 = \frac{\alpha\beta\gamma}{6}$

Hence, locus of $P\ (\alpha, \beta, \gamma)$ is $xyz = 6\,k^3$.

24 The required line passes through the point $\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ and is perpendicular to the lines

$\mathbf{r} = (\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + \lambda\,(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

and $\mathbf{r} = (2\mathbf{i} + 6\mathbf{j} + \mathbf{k}) + \mu\,(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

So, it is parallel to the vector.

$\therefore\ \mathbf{b} = (2\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$

$= (\mathbf{i} - 5\mathbf{j} + 3\mathbf{k})$

The required equation is

$\mathbf{r} = (\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + \lambda\,(\mathbf{i} - 5\mathbf{j} + 3\mathbf{k})$

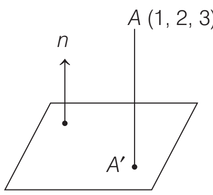
25 Let $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

Line through A and parallel to $\mathbf{n}$ is

$\mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} + \lambda(3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

$= (3\lambda + 1)\mathbf{i} + (2 - \lambda)\mathbf{j} + (3 + 4\lambda)\mathbf{k} \quad \ldots(i)$

Eq. (i) must satisfy the plane $3x - y + 4z = 0$.



$\therefore \quad 3(3\lambda + 1) - (2 - \lambda) + 4(3 + 4\lambda) = 0$

$\Rightarrow \quad 26\lambda + 13 = 0$

$\Rightarrow \quad \lambda = -\frac{1}{2}$

Hence, A' is $\left(-\frac{1}{2}, \frac{5}{2}, 1\right)$ which is the foot of the perpendicular from A on the given plane.

26 DR's of OC are $(1, -1, 2)$.

Let the equation of plane passing through $(1, 0, 1)$ is

$a(x - 1) + b(y - 0) + c(z - 1) = 0 \quad \ldots(i)$

Since, its normal is perpendicular to OC

$\therefore \quad 1\cdot a + (-1)b + 2c = 0$

$\Rightarrow \quad a - b + 2c = 0 \quad \ldots(ii)$

As Eq. (i) passes through (3, 1, 2).

$\therefore \quad 2a + b + c = 0 \quad \ldots(iii)$

$\Rightarrow \quad \frac{a}{-1} = \frac{b}{1} = \frac{c}{1}$

[from Eqs. (ii) and (iii)]

Hence, required equation of plane be $x - y - z = 0$.

27 Let DC's of PQ be l, m and n.

$\therefore\ 3l - m + n = 0$ and $5l + m + 3n = 0$

$\therefore \quad \frac{l}{-3 - 1} = \frac{m}{5 - 9} = \frac{n}{3 + 5}$

$\Rightarrow \quad \frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$

Thus, the equation of plane perpendicular to PQ will have $x + y - 2z = \lambda$.

It passes through (2, 1, 4), therefore $\lambda = -5$.

Hence, the required equation of plane be $x + y - 2z = -5$

28 The line of intersection of the planes $\mathbf{r}\cdot(3\mathbf{i} - \mathbf{j} + \mathbf{k}) = 1$ and $\mathbf{r}\cdot(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) = 2$ is perpendicular to each of the normal vectors.

Here, $\mathbf{n}_1 = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$ and

$\mathbf{n}_2 = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$\therefore$ It is parallel to the vector $\mathbf{n}_1 \times \mathbf{n}_2$

$= (3\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$

$= -2\mathbf{i} + 7\mathbf{j} + 13\mathbf{k}$

29 Since, lines are $\frac{x}{1} = \frac{y + a}{1} = \frac{z}{1}$

and $\frac{x + a}{2} = \frac{y}{1} = \frac{z}{1}$

Let $P \cong (r, r - a, r)$ and $Q \cong (2\lambda - a, \lambda, \lambda)$ be the points of I and II lines.

So, DR's of PQ are

$r - 2\lambda + a, r - \lambda - a, r - \lambda$.

According to the given question,

$$\frac{r - 2\lambda + a}{2} = \frac{r - \lambda - a}{1} = \frac{r - \lambda}{2}$$

From I and II terms, $r - a = 2a \Rightarrow r = 3a$

From II and III terms, $\lambda = a$

$\therefore\ P \equiv (3a, 2a, 3a)$ and $Q \equiv (a, a, a)$

30 Let $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j} - 6\,\mathbf{k}$ and

$\mathbf{b} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$

$\because$ Diagonals of a parallelogram in terms of its sides are

$\mathbf{p} = \mathbf{a} + \mathbf{b}$ and $\mathbf{q} = \mathbf{b} - \mathbf{a}$

$\Rightarrow\ \mathbf{p} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{q} = -\mathbf{i} - 7\mathbf{j} + 11\mathbf{k}$

The unit vectors along the diagonals are

$$\frac{5\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{25 + 1 + 1}} \text{ and } \frac{-\mathbf{i} - 7\mathbf{j} + 11\mathbf{k}}{\sqrt{(-1)^2 + 49 + 121}}$$

$$\Rightarrow \frac{5\mathbf{i} + \mathbf{j} - \mathbf{k}}{\sqrt{27}} \text{ and } \frac{-\mathbf{i} - 7\mathbf{j} + 11\mathbf{k}}{\sqrt{171}}$$

31 Since, $\mathbf{p}$ and $\mathbf{q}$ are perpendicular.

$\therefore \quad \mathbf{p}\cdot\mathbf{q} = 0$

Let $\quad \mathbf{q} = x\mathbf{i} + y\mathbf{j}$, then

$(8\mathbf{i} + 6\mathbf{j})(x\mathbf{i} + y\mathbf{j}) = 0 \Rightarrow 8x + 6y = 0$

$\therefore \quad y = -\frac{8x}{6} = -\frac{4x}{3}$

$$\mathbf{q} = x\mathbf{i} + \left(\frac{-4x}{3}\right)\mathbf{j} = \frac{3x\mathbf{i} - 4x\mathbf{j}}{3}$$

$$= \frac{x}{3}(3\mathbf{i} - 4\mathbf{j})$$

Again, the projection of vector $\mathbf{r} = \pm(x_1\,\mathbf{i} + x_2\,\mathbf{j})$ on vector $\mathbf{p}$ is 2 and on $\mathbf{q}$ is 4.

$\therefore\ 2 = \left|\frac{8x_1 + 6x_2}{10}\right|$ and $4 = \left|\frac{6x_1 - 8x_2}{10}\right|$

$\Rightarrow \quad 8x_1 + 6x_2 = 20$

and $\quad 6x_1 - 8x_2 = 40$

$\Rightarrow \quad 4x_1 + 3x_2 = 10$

and $\quad 3x_1 - 4x_2 = 20$

$\Rightarrow \quad x_1 = 4$ and $x_2 = -2$

$\therefore \quad \mathbf{r} = \pm(4\mathbf{i} - 2\mathbf{j}) = \pm\,2\,(2\mathbf{i} - \mathbf{j})$

32 The plane is

$(2 + 3\lambda)x + (\lambda - 1)y + (\lambda + 1)z - 5\lambda - 3 = 0$

Its distance from $(2, 1, -1)$ is $\frac{1}{\sqrt{6}}$.

$$\therefore \frac{(4 + 6\lambda + \lambda - 1 - \lambda - 1 - 5\lambda - 3)^2}{(2 + 3\lambda)^2 + (\lambda - 1)^2 + (\lambda + 1)^2} = \frac{1}{6}$$

$\Rightarrow (5\lambda + 24)\lambda = 0 \Rightarrow \lambda = \frac{-24}{5}$ or 0

The planes are $2x - y + z - 3 = 0$

and $\ 62x + 29y + 19z - 105 = 0$

33 Equation of a plane passing through the point $(2, -1, -3)$ and parallel to the given line is

$$\begin{vmatrix} x - 2 & y + 1 & z + 3 \\ 3 & 2 & -4 \\ 2 & -3 & 2 \end{vmatrix} = 0$$

$\Rightarrow \quad (x - 2)(4 - 12) - (y + 1)(6 + 8) + (z + 3)(-9 - 4) = 0$

$\Rightarrow \quad 8x + 14y + 13z + 37 = 0$

34 The common normal is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$\mathbf{r} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\therefore$ Shortest distance $= (\mathbf{j} + \mathbf{k})\cdot\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{\sqrt{6}}$

35 Normal to OAB is $\mathbf{OA} \times \mathbf{OB}$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$$

Normal to ABC is

$$\mathbf{AB} \times \mathbf{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \mathbf{i} - 5\mathbf{j} - 3\mathbf{k}$$

If θ is angle between the planes, then

$$\cos\theta = \frac{5+5+9}{\sqrt{35}\cdot\sqrt{35}} = \frac{19}{35}$$

36 Let $\mathbf{r} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$

Since, $\mathbf{r}\perp\mathbf{p}$ and $\mathbf{r}\perp\mathbf{q}$

$\therefore\quad \mathbf{r}\cdot\mathbf{p} = 0$ and $\mathbf{r}\cdot\mathbf{q} = 0$

$\Rightarrow\quad 2a_1 - 3a_2 + 3a_3 = 0$

and $\quad 4a_1 - 2a_2 + a_3 = 0$

$\Rightarrow\quad 2a_1 - 3a_2 + 3a_3 = 0$

and $\quad 4a_1 - 2a_2 + a_3 = 0$

$$\therefore\quad \frac{a_1}{-3-(-6)} = \frac{-a_2}{12-2} = \frac{a_3}{-4-(-12)} = k \quad \text{[say]}$$

$$\Rightarrow\quad \frac{a_1}{3} = \frac{a_2}{10} = \frac{a_3}{8} = k$$

$\therefore\quad a_1 = 3k,\ a_2 = 10k,\ a_3 = 8k \quad$ ...(i)

Again, $(a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k})$

$(2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = -12$

$\Rightarrow\quad 2a_1 - 4a_2 + 2a_3 = -12 \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$6k - 4(10k) + 2(8k) = -12$

$6k - 40k + 16k = -12$

$$-18k = -12 \Rightarrow k = \frac{12}{18} = \frac{2}{3}$$

$$\therefore\quad a_1 = 2, a_2 = \frac{20}{3}, a_3 = \frac{16}{3}$$

$$\therefore\quad \mathbf{r} = 2\mathbf{i} + \frac{20}{3}\mathbf{j} + \frac{16}{3}\mathbf{k}$$

37 Plane passing through a point (x_1, y_1, z_1) is

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

$\therefore$ Plane through $(2, 0, 0)$ is

$a(x-2) + b(y-0) + c(z-0) = 0 \quad$ ...(i)

contains $(0, 2, 0)$, if

$-2a + 2b = 0 \Rightarrow -a + b = 0 \quad$ ...(ii)

Since, plane

$a(x-2) + b(y-0) + c(z-0) = 0$

makes an angle $\frac{\pi}{3}$ with the plane

$2x + 3y = 5$.

$$\therefore\quad \cos\frac{\pi}{3} = \frac{2a+3b}{\sqrt{a^2+b^2+c^2}\sqrt{4+9}}$$

$$\Rightarrow\quad \frac{1}{2} = \frac{2a+3b}{\sqrt{(a^2+b^2+c^2)(13)}}$$

$$\Rightarrow\quad \frac{1}{4} = \frac{(2a+3b)^2}{[\sqrt{(a^2+b^2+c^2)(13)}]^2}$$

$\Rightarrow\quad 13(a^2 + b^2 + c^2) = 100a^2$

$\because\quad a = b$

$\therefore\quad 13(2a^2 + c^2) = 100a^2$

$\Rightarrow\quad 26a^2 + 13c^2 = 100a^2$

$$\Rightarrow\quad 13c^2 = 74a^2 \Rightarrow c = \sqrt{\frac{74}{13}}\,a$$

$$\therefore\quad a : b : c = a : a : \sqrt{\frac{74}{13}}\,a$$

$= a : a : \sqrt{5.7}\,a$

$= 1 : 1 : \sqrt{5.7}$

38 Shortest distance

$$= \frac{\left\|\begin{matrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\|}{\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{matrix}\right\|}$$

$$= \frac{\left\|\begin{matrix} -1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 4 \end{matrix}\right\|}{\left\|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 2 & 3 & 4 \end{matrix}\right\|}$$

$$= \frac{|-1(8-6) + 1(4-12) + 1(9-4)|}{|\mathbf{i}(8-6) + \mathbf{j}(4-12) + \mathbf{k}(9-4)|}$$

$$= \left|\frac{-2-8+5}{\sqrt{93}}\right|$$

$$= \frac{5}{\sqrt{93}} = 0.52 \text{ unit}$$

39 Any point on the lines L_1 and L_2 are $(-3r_1 - 1, 2r_1 + 3, r_1 - 2)$ and $(r_2, -3r_2 + 7, 2r_2 - 7)$

Since, they intersect each other, therefore

$-3r_1 - 1 = r_2, 2r_1 + 3 = -3r_2 + 7$

and $\quad r_1 - 2 = 2r_2 - 7$

On solving, we get

$r_2 = 2$ and $r_1 = -1$

Hence, the required point is $(2, 1, -3)$.

40 Let θ be an angle between unit vectors $\mathbf{a}$ and $\mathbf{b}$. Then, $\mathbf{a}\cdot\mathbf{b} = \cos\theta$

Now, $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}$

$$= 2 + 2\cos\theta = 4\cos^2\frac{\theta}{2}$$

$$\Rightarrow\quad |\mathbf{a} + \mathbf{b}| = 2\cos\frac{\theta}{2}\ |\mathbf{a} - \mathbf{b}| = 2\sin\frac{\theta}{2}$$

$\Rightarrow\quad |\mathbf{a} + \mathbf{b}| + |\mathbf{a} - \mathbf{b}|$

$$= 2\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) \le 2\sqrt{2}$$

41 Let $\alpha = \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \beta = \lambda\mathbf{b} + 4\mathbf{c}$

and $\gamma = (2\lambda - 1)\mathbf{c}$

$$\text{Then, } [\alpha\ \beta\ \gamma] = \begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & (2\lambda - 1) \end{vmatrix} [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$[\alpha\ \beta\ \gamma] = \lambda(2\lambda - 1)[\mathbf{abc}]$

$\Rightarrow\quad [\alpha\ \beta\ \gamma] = 0,$

If $\lambda = 0, \frac{1}{2} \quad [\because [\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \ne 0]$

Hence, α, β and γ are non-coplanar for all value of λ except two values 0 and $\frac{1}{2}$.

42 Since, $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c} = 0, \mathbf{b}\cdot\mathbf{c} = \frac{1}{2}$

$\therefore\quad |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}\times\mathbf{c}| = 1$

Now, $|\mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c}|^2 = |\mathbf{a}\times\mathbf{b}|^2 + |\mathbf{a}\times\mathbf{c}|^2 - 2(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{a}\times\mathbf{c})$

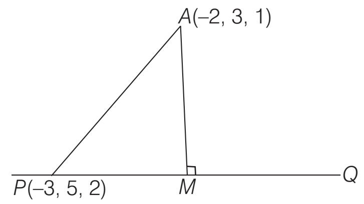
$$= 1 + 1 - 2\begin{vmatrix} 1 & 0 \\ 0 & 1/2 \end{vmatrix} = 1$$

43 Here, $\alpha = \beta = \gamma$

$\because\quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\therefore\quad \cos\alpha = \frac{1}{\sqrt{3}}$$

$$\text{DC's of } PQ \text{ are } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$



PM = Projection of AP on PQ

$$= \left|(-2+3)\frac{1}{\sqrt{3}} + (3-5)\cdot\frac{1}{\sqrt{3}} + (1-2)\cdot\frac{1}{\sqrt{3}}\right| = \frac{2}{\sqrt{3}}$$

and

$AP = \sqrt{(-2+3)^2 + (3-5)^2 + (1-2)^2}$

$= \sqrt{6}$

$$AM = \sqrt{(AP)^2 - (PM)^2} = \sqrt{6 - \frac{4}{3}} = \sqrt{\frac{14}{3}}$$

44 Equation of the plane through $(5, 1, 2)$ is

$a(x-5) + b(y-1) + c(z-2) = 0 \quad$ ...(i)

Given plane (i) is perpendicular to the line

$$\frac{x-2}{1/2} = \frac{y-4}{1} = \frac{z-5}{1} \quad \text{...(ii)}$$

$\therefore$ Equation of normal of Eq. (i) and straight line (ii) are parallel

$$\text{i.e.}\quad \frac{a}{1/2} = \frac{b}{1} = \frac{c}{1} = k \text{ (say)}$$

$$\therefore\quad a = \frac{k}{2}, b = k, c = k$$

From Eq. (i),

$$\frac{k}{2}(x-5) + k(y-1) + k(z-2) = 0$$

$\Rightarrow\quad x + 2y + 2z = 11$

Any point on Eq. (ii) is

$$\left(2 + \frac{\lambda}{2}, 4 + \lambda, 5 + \lambda\right)$$

which lies on Eq. (iii), then $\lambda = -2$.

$\therefore$ Required point is $(1, 2, 3)$.

45 If the lines have a common point, then there exists λ and μ such that

$6 - \lambda = 1 + \mu$

and $\quad -1 + 2\lambda = -1 + 3\mu$

$\Rightarrow\quad \lambda = 3, \mu = 2$

$\therefore\quad \mathbf{r} = 3\mathbf{a} + 5\mathbf{c}$

46 In an isosceles ΔABC in which $AB = AC$ the median and bisector from A must be same line, so Statement II is true.

Now, $\mathbf{AD} = \dfrac{\mathbf{u} + \mathbf{v}}{2}$

$\Rightarrow \quad |\mathbf{AD}|^2 = \dfrac{1}{4}[|\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u} \cdot \mathbf{v}]$

$= \dfrac{1}{4}(2 + 2\cos\alpha) = \cos^2 \dfrac{\alpha}{2}$

$\therefore$ Unit vector along

$\mathbf{AD} = \dfrac{1}{2\cos\dfrac{\alpha}{2}}(\mathbf{u} + \mathbf{v})$

47 Direction ratios of PQ are $6 - 2, 5 - 1, -2 - 6$ i.e. $4, 4, -8$ which are proportional to the direction ratios of the normal to the plane π, so PQ is perpendicular to π.

Hence, Statement I is false and Statement II is true.

48 The equation of plane through the line of intersection of the planes $x + y + z = 6$ and $2x + 3y + 4z + 5 = 0$ is $(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$ ...(i)

Since, it passes through $(4, 4, 4)$, then

$(4 + 4 + 4 - 6) + \lambda(8 + 12 + 16 + 5) = 0$

$\Rightarrow \quad 6 + 41\lambda = 0 \Rightarrow \lambda = -\dfrac{6}{41}$

From Eq. (i), we get

$41(x + y + z - 6) - 6(2x + 3y + 4z + 5) = 0$

$\therefore \; 29x + 23y + 17z = 276$

49 Adjacent sides of a parallelogram are given as $\mathbf{a} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$ and $\mathbf{b} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$.

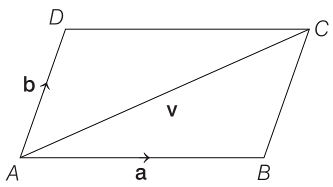
Then, the diagonal of a parallelogram is given by $\mathbf{v} = \mathbf{a} + \mathbf{b}$.

[since, from the figure, it is clear that resultant of adjacent sides of a parallelogram is given by the diagonal]

$\therefore \; \mathbf{v} = 2\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} + \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

$= (2 + 1)\mathbf{i} + (-4 - 2)\mathbf{j} + (5 - 3)\mathbf{k}$

$= 3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$



$\therefore \; |\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{(3)^2 + (-6)^2 + (2)^2}$

$= \sqrt{9 + 36 + 4}$

$= \sqrt{49} = 7$

Thus, the unit vector parallel to the diagonal is

$\dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{3\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}}{7}$

$= \dfrac{3}{7}\mathbf{i} - \dfrac{6}{7}\mathbf{j} + \dfrac{2}{7}\mathbf{k}$

Also, area of parallelogram

$ABCD = |\mathbf{a} \times \mathbf{b}| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$

$= |\mathbf{i}(12 + 10) - \mathbf{j}(-6 - 5) + \mathbf{k}(-4 + 4)|$

$= |22\mathbf{i} + 11\mathbf{j} + 0\mathbf{k}|$

$= \sqrt{(22)^2 + (11)^2 + 0^2}$

$= \sqrt{(11)^2 + (2^2 + 1^2)}$

$= 11\sqrt{5}$ sq units